Supply and demand model with differential equation

Given the model:

$$D(t) = a + b \cdot p(t)$$
$$S(t) = a_1 + b_1 \cdot \dot{p}(t)$$
$$\dot{p}(t) = p(t) + c \cdot p'$$
$$D(t) = S(t)$$

1. Solve the problem.
2. Analyze the stability of the model assuming we are in the normal case where the Demand has a negative slope, while the Supply has a positive slope. Evaluate it for both $c > 0$ and $c < 0$.

Solution

1. We equate supply and demand and insert the price:

$$a + bp = a_1 + b_1(p + cp')$$

Rearranging:

$$\begin{aligned} a + bp &= a_1 + b_1p + b_1cp' \\ -b_1cp' + p(b - b_1) &= a_1 - a \\ -b_1c\frac{dp}{dt} + p(b - b_1) &= a_1 - a \\ -b_1c\frac{dp}{dt} &= a_1 - a - p(b - b_1) \\ \frac{dp}{a_1 - a - p(b - b_1)} &= \frac{-dt}{b_1c} \end{aligned}$$

We integrate both sides. On the left side:

$$\int \frac{dp}{a_1 - a - p(b - b_1)}$$

We use substitution:

$$\begin{aligned} u &= a_1 - a - p(b - b_1) \\ du &= (-b + b_1)dp \\ \frac{du}{(-b + b_1)} &= dp \end{aligned}$$

$$\int \frac{1}{u}\frac{du}{(-b + b_1)} = \frac{1}{(-b + b_1)}\int \frac{du}{u} = \frac{\ln(u)}{(-b + b_1)} + C = \frac{\ln(a_1 - a - p(b - b_1))}{(-b + b_1)} + C$$

Returning to the differential equation:

$$\frac{\ln(a_1 - a - p(b - b_1))}{(-b + b_1)} + C = \frac{-t}{b_1c}$$

Solving for p:

$$\begin{aligned} \ln(a_1 - a - p(b - b_1)) &= \frac{t(b - b_1)}{b_1c} + K \\ a_1 - a - p(b - b_1) &= e^{\frac{t(b-b_1)}{b_1c}}T \\ -p(b - b_1) &= e^{\frac{t(b-b_1)}{b_1c}}T - a_1 + a \\ p &= e^{\frac{t(b-b_1)}{b_1c}}A + \frac{a - a_1}{b_1 - b} \end{aligned}$$

2. We take the limit as $t \to \infty$. We know that $b_1 > 0$ and that $b < 0$, then $\frac{(b-b_1)}{b_1} < 0$. If $c > 0$ then: $\frac{(b-b_1)}{cb_1} < 0$ and the system is stable tending to:

$$\lim_{t\to\infty} p = \frac{a - a_1}{b_1 - b}$$

If $c < 0$ then: $\frac{(b-b_1)}{cb_1} > 0$ and the system is unstable tending to:

$$\lim_{t\to\infty} p = \infty$$